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Coclass and cohomology

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Dedicated to Claus Michael Ringel on the occasion of his 60th birthday.

Abstract

Recently, it was proved by Leedham-Green and others that with a finite number of exceptions, every p-group of coclass r is a quotient of one of only a finite number of p-adic uniserial space groups. In this paper we use that structure to demonstrate that there are only finitely many isomorphism classes of cohomology rings of 2-groups of coclass r with coefficients in any fixed field k of characteristic 2. In addition, there is experimental evidence indicating that in many cases successive quotients of the uniserial space groups have isomorphic cohomology rings.

MSC: 20J06

1. Introduction

This paper considers what the coclass classification of finite p-groups might say about the cohomology rings of groups. The classification was recently completed by Leedham-Green and others (see [7]). It says roughly that, with a finite number of exceptions, every p-group having coclass r is associated to one of a finite number of p-adic uniserial space groups. The nature of the association for 2-groups is given in detail in Theorem 5.2. For p-groups with p odd, the association is somewhat more complicated and we refer the reader

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to [7] for details. The coclass of a p-group G is $r = n - c$, where p^n is the order of G and c is the class of G, the length of the lower central series of G.

The model for the association is the situation with 2-groups of coclass 1. With exactly two exceptions, every 2 group of coclass 1 is one of the dihedral groups, semi-dihedral groups or (generalized) quaternion groups. Each of these is associated to the infinite dihedral group $\langle z, y \mid z^2 = 1, zyz^{-1} = y^{-1} \rangle$ in the sense that it isomorphic either to a quotient of that group or to a central extension of a quotient of that group by a cyclic group of order 2. Thus, there is a unique family of 2-groups of coclass 1.

It has long been known that if k is a field of characteristic 2 and if G_1 and G_2 are dihedral 2-groups, then the cohomology rings $\mathrm{H}^*(G_1, k)$ and $\mathrm{H}^*(G_2, k)$ are isomorphic [4]. Similarly, the cohomology rings of any two quaternion groups are isomorphic [4] and the cohomology ring of any two semi-dihedral groups are isomorphic [5]. Hence, among all of the groups of coclass 1 there is only a finite number of isomorphism types of cohomology rings. The main theorem of this paper shows that this is correct in general. That is, for any r there are only finitely many isomorphism classes of cohomology rings of 2-groups of coclass r with coefficients in a fixed field k of characteristic 2. We should note that $\mathrm{H}^*(G, k) \cong k \otimes_{\mathbb{F}_2} \mathrm{H}^*(G, \mathbb{F}_2)$ is really defined over the prime field. Hence, the choice of the field k is largely irrelevant in this consideration.

In fact, there is experimental evidence indicating that a lot more is true. The proof of the theorem shows only that the number is finite, without giving any sort of effective bounds. The proof is mostly a collection of counting arguments on the number of possible outcomes of certain spectral sequences. It does not actually say that any two groups have isomorphic cohomology rings. The computational evidence indicates that successive quotients of the uniserial space groups do indeed have isomorphic cohomology rings. The evidence is somewhat scant because of the limitations of the computational methods. However, there is enough to be intriguing.

For these reasons, this paper should be seen as a first step in an investigation of the isomorphisms between cohomology rings. The methods for calculating the rings are not yet sufficiently refined to determine what other theorems might be provable. For one thing, a similar theorem should be true for p-groups in the case that p is an odd prime. Also, similar results have been found in the studies [2,13], which though much more specialized, give much stronger outcomes.

Following this introduction, the next two sections of the paper develop some technicalities of counting with spectral sequences. This is followed by an investigation of the cohomology ring of the quotients of a particularly important 2-adic space group. The proof of the main theorem is found in Section 5. The last section discusses the case of cohomology rings of p-groups for odd p as well as reviewing some of the experimental evidence.

Except as noted, all of the computer calculations of the last section were run using the MAGMA computer algebra package [1].

2. Ungrading the spectral sequence

In this section we investigate the extent to which a ring may differ from a graded version of itself. The main interest here is in the difference between a cohomology ring $\mathrm{H}^*(G, k)$

of a group G and the graded version which is the E_∞ page of a spectral sequence

$$E_2^{r,s} = \mathrm{H}^r(G/H, \mathrm{H}^s(H,k)) \Rightarrow \mathrm{H}^*(G,k)$$

of the group extension, where H is a normal subgroup of G. We write the theorem in more general terms. In the above situation it tells us that given the ring $E_\infty^{*,*}$, there is only a finite number of ways of creating $\mathrm{H}^*(G,k)$.

Suppose that $R = \oplus_{i \geqslant 0} R^i$ is a graded-commutative, finitely generated k-algebra over a finite field k. Assume that $R^0 = k$. Then each R^i is finite dimensional over k. Assume that R has a filtration $\mathscr{F}^*$ such that $\mathscr{F}^j(R^i) = 0$ whenever $j > i$. Assume also that the filtration is multiplicative in the sense that $\mathscr{F}^i(R)\mathscr{F}^j(R) \subseteq \mathscr{F}^{i+j}(R)$ for all i and j.

Let $S = \sum_{i,j \geqslant 0} S^{i,j}$ be the doubly graded algebra obtained from R by letting

$$S^{i,j} = \mathscr{F}^i(R^{i+j})/\mathscr{F}^{i+1}(R^{i+j}).$$

The multiplication on S is given by the following formula. If $x \in \mathscr{F}^i(R^m)$ and $y \in \mathscr{F}^j(R^n)$, then

$$(x + \mathscr{F}^{i+1}(R^m))(y + \mathscr{F}^{j+1}(R^n)) = xy + \mathscr{F}^{i+j+1}(R^{m+n}).$$

Theorem 2.1. *Let R be a finitely generated, graded-commutative k-algebra as above and let S be the doubly graded ring defined by a filtration $\mathscr{F}^*$ on R. Assume that the field k is finite. Then the structure of R is determined by the structure of S within a finite number of possibilities.*

Proof. Suppose that $s_1, \ldots, s_t$ is a set of homogeneous generators for the ring S. For each i, there is a pair of non-negative integers (j_i, k_i) such that $s_i \in S^{j_i,k_i}$. For each i, let $r_i \in R^{j_i+k_i}$ be any element such that $r_i + \mathscr{F}^{j_i}(R) = s_i$. We claim first that $r_1, \ldots, r_t$ generate R.

For suppose that $r \in \mathscr{F}^i(R^n), r \notin \mathscr{F}^{i+1}(R^n)$. Then we know that $r + \mathscr{F}^{i+1} = f(s_1, \ldots, s_t)$ for some polynomial f (in graded commuting variables). Then we have that $r - f(r_1, \ldots r_t) \in \mathscr{F}^{i+1}(R^n)$. Now we continue along this line until we get that $r - g(r_1, \ldots, r_t) = 0$ for some polynomial g. This establishes the claim.

Let $F = k\langle\alpha_1, \ldots, \alpha_t\rangle$ be a free graded-commutative algebra such that we have a graded homomorphism $\psi : F \longrightarrow S$. Note that this requires that the degree of α_i be the same as that of s_i for all i. The variables $\alpha_1, \ldots, \alpha_t$ satisfy the relations $\alpha_i\alpha_j = (-1)^{\deg(\alpha_i)\deg(\alpha_j)}\alpha_j\alpha_i$, and these are essentially the only relations on the variables. Of course, if $p = 2$, then F is a polynomial ring. In any case, every element of F is a sum of monomials $\alpha_1^{u_1}\alpha_2^{u_2}\cdots\alpha_t^{u_t}$, where if $p > 2$, the exponent u_i is either 0 or 1 when the degree of α_i is odd (since $\alpha_i^2 = 0$ in such a case). Moreover, we can regard F as a doubly graded ring and ψ as map of doubly graded rings by assigning to each α_i the double degree (j_i, k_i).

Because F is a free object in the category of graded commutative k-algebras, there is a homomorphism $\theta : F \longrightarrow R$ given by $\theta(\alpha_i) = r_i$ for all i. In particular, note that the degrees match up and this is a graded homomorphism. From the above argument, we have that θ is surjective. Let $\mathscr{I}$ denote the kernel of ψ, and suppose that $f_1, \ldots, f_n$ is a minimal set of doubly homogeneous generators for $\mathscr{I}$. For each i, let (a_i, b_i) be the double degree of $f_i = f_i(\alpha_1, \ldots, \alpha_t)$. It follows that $f_i(r_1, \ldots, r_t) \in \mathscr{F}^{a_i+1}(R^{a_i+b_i})$. Hence there exist

$f_{i,1}, \ldots, f_{i,b_i}$ in F having homogeneous (double) degrees $(a_i+1, b_i-1), \ldots, (a_i+b_i, 0)$, respectively, such that

$$f_i(r_1, \ldots, r_t) + \sum_{k=1}^{b_i} f_{i,k}(r_1, \ldots, r_t) = 0.$$

Hence, if $g_i = f_i + \sum f_{i,k}$, then g_i is in the kernel $\mathscr{K}$ of θ. Let $\mathscr{J}$ be the ideal of F generated by $g_1, \ldots, g_n$. Clearly $\mathscr{J} \subseteq \mathscr{K}$. We claim that $\mathscr{J} = \mathscr{K}$.

Suppose that $h = h(\alpha_1, \ldots, \alpha_t)$ is a homogeneous polynomial of degree m that is contained in $\mathscr{K}$, the kernel of θ. Write $h = h_\ell + h_{\ell+1} + \cdots + h_m$, where h_j is of homogeneous double degree $(j, m-j)$ and ℓ is some integer between 0 and m with $h_\ell \neq 0$. Then

$$h_\ell(s_1, \ldots, s_t) = h(r_1, \ldots, r_t) + \mathscr{F}^{\ell+1}(R^m) = 0.$$

We must have that $h_\ell \in \mathscr{I}$ and hence that $h_\ell = \sum_{k=1}^n \beta_k f_k$, where β_k is homogeneous of double degree $(\ell - a_k, m - \ell - b_k)$. Then we get that $h - \sum_k \beta_k g_k$ is in $\mathscr{K}$ and moreover, $h - \sum_k \beta_k g_k$ is a sum of doubly homogeneous polynomials in double degrees $(u, m-u)$ where $u \geqslant \ell$. Now perform this operation again with h replaced by $h' = h - \sum_k \beta_k g_k$. After a finite number of steps we have $h \in \mathscr{J}$ as desired.

So we have shown that $R \cong F/\mathscr{J}$. The proof of the theorem is a consequence of the fact that $\mathscr{J} = (g_1, \ldots, g_n)$ and for each i, $g_i = f_i + u_i$ where u_i has the same total degree as g_i but higher double degree. That is, R differs from S only in the choices of the elements $u_1, \ldots, u_n$, and there are only finitely many possibilities for each such choice. We leave it as an exercise to the reader to show that the number of possibilities does not depend on our choice of the generators. $\square$

Some of the steps in the proof of the theorem can be summarized in the following results which are useful to us later in the paper. The results are corollaries to the proof and we do not repeat the arguments.

Corollary 2.2. *Suppose that we have an extension* $1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$ *of finite groups. Assume that the structure of the* E_∞ *page of the spectral sequence of the extension is known either as a ring or as a finitely generated module over some homogeneous set of parameters in* $E_\infty^{*,0} \cup E_\infty^{0,*}$. *Then the structure of* $\mathrm{H}^*(G,k)$ *as a ring is determined up to a finite number of possibilities.*

Corollary 2.3. *Suppose that the structure of* $\mathrm{H}^*(G,k)$ *is known as a finitely generated module over some homogeneous finitely generated subring. Then the ring structure of* $\mathrm{H}^*(G,k)$ *is determined up to a finite number of possibilities.*

3. The counting theorems

The purpose of this section is to prove some theorems on the number of cohomology rings that can be built from an extension. Another way of stating the proposition below is that if G is a p-group with a normal subgroup H, the cohomology ring $\mathrm{H}^*(G,k)$ is determined by $\mathrm{H}^*(H,k)$ and $\mathrm{H}^*(G/H,k)$ within a finite number of possibilities provided the action of G/H on H is trivial and it is known that the spectral sequence

$$E_2^{r,s} = \mathrm{H}^r(G/H, \mathrm{H}^s(H,k)) \Rightarrow \mathrm{H}^{r+s}(G,k)$$

collapses after some bounded number of pages. In particular, we need not know anything about the groups H and G/H, only about their cohomology rings and the action of G/H on H. We suspect that the second condition in the proposition is not necessary.

Proposition 3.1. *Suppose that S and T are finitely generated k-algebras and that b is a positive integer. Then there are only finitely many k-algebras R such that* $R \cong \mathrm{H}^*(G,k)$ *where G is a finite p-group satisfying the following the properties*:

(1) *G has a normal subgroup H such that* $\mathrm{H}^*(H,k) \cong S$ *and* $\mathrm{H}^*(G/H,k) \cong T$,
(2) G/H *has trivial action on H, and*
(3) *the spectral sequence of the group extension*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

stops (*collapses*) *on or before page b* (*i.e.* $E_b^{*,*} = E_\infty^{*,*}$).

Proof. Suppose that G and H satisfy the conditions of the theorem. Because G/H has trivial action on H, $E_2^{r,s} \cong \mathrm{H}^r(G/H,k) \otimes \mathrm{H}^s(H,k)$, and moreover, the isomorphism is an isomorphism of rings. Now choose a complete homogeneous set of parameters $\gamma_1, \ldots, \gamma_u$ for $\mathrm{H}^*(H,k)$. That is, $\gamma_1, \ldots, \gamma_u$ generate a polynomial subring over which $\mathrm{H}^*(H,k)$ is a finitely generated module. For convenience of notation we identify $\mathrm{H}^*(H,k)$ with the subalgebra $\mathrm{H}^*(H,k) \otimes 1 \subseteq E_2^{*,*}$.

For an element γ of even degree s in $\mathrm{H}^*(H,k)$, suppose that γ survives to the E_n page of the spectral sequence. If $d_n(\gamma) = \mu \in E_n^{n,s-n+1}$, then because d_n is a derivation we have $d_n(\gamma^p) = p \cdot d_n(\gamma)\gamma^{p-1} = p\mu\gamma^{p-1} = 0$. Because the spectral sequence stops at the E_b page, we must have that for all i, $\zeta_i = \gamma_i^{p^b}$ is a universal cycle. That is, ζ_i is in the kernel of every one of the differentials d_j for $j \geqslant 2$.

Let $\zeta_{u+1}, \ldots, \zeta_v$ be a homogeneous set of parameters for $\mathrm{H}^*(G/H,k)$ which we identify with the subalgebra $E_2^{*,0}$ of $E_2^{*,*}$. Then $E_2^{*,*}$ is a finitely generated module over the polynomial subalgebra W generated by $\zeta_1, \ldots, \zeta_v$. Next we note inductively that, because $d_j(\zeta_i) = 0$, each d_j is W-module homomorphism. Therefore, each $E_{j+1}^{*,*}$ is a finitely generated W-module. The generators can be chosen to be homogeneous. The key point is that if $\alpha_1, \ldots, \alpha_q$ is a set of generators for $E_j^{*,*}$ as a W-module, then the homomorphism d_j is determined entirely by the choice of the images $d_j(\alpha_1), \ldots, d_j(\alpha_q)$. For each i there is only a finite number of choices of the image $d_j(\alpha_i)$ simply because $E_j^{\ell_1+j,\ell_2-j+1}$ is a finite set, (with (ℓ_1, ℓ_2) being the degree of α_i). Consequently, for each $j \geqslant 2$ there is only a finite number of possibilities for the map d_j and for the structure of $E_{j+1}^{*,*}$ as both a W-module and as a ring. Because $d_j = 0$ for $j \geqslant b$, there are only finitely many possibilities for $E_\infty^{*,*}$.

The fact that there are only finitely many possibilities for R follows from Theorem 2.1. $\square$

Next, we consider the case that H has bounded order. It is sufficient to consider the situation in which $H \cong C_p$ is cyclic of order p and apply induction. What we need is the following.

Lemma 3.2. *Suppose that $H \cong C_p$ is contained in an abelian subgroup A such that $|G : A| = p^n$. Then the spectral sequence $E_2^{r,s} = \mathrm{H}^r(G/H, \mathrm{H}^s(C_p, k)) \Rightarrow \mathrm{H}^{r+s}(G, k)$ stops at the $p^n + 1$ page.*

Proof. There exists an element $\eta \in \mathrm{H}^2(A, k)$ such that the restriction from A to H of η is not zero. This is because A is abelian and we know the cohomology of abelian p-groups. So let $\zeta = \mathcal{N}\mathrm{orm}_A^G(\eta)$, where $\mathcal{N}\mathrm{orm}_A^G$ is the Evens Norm Map. The usual Mackey formula for norms (see [3, Theorem 6.3.5.3]) tells us that $\mathrm{res}_{G,H}(\zeta) \neq 0$. The restriction map on cohomology from G to H is the edge homomorphism on the spectral sequence. So the image of the restriction $\mathrm{res}_{G,H} : \mathrm{H}^{2p^n}(G, k) \longrightarrow \mathrm{H}^{2p^n}(H, k)$ is isomorphic to $E_\infty^{0,2p^n}$. Let $\zeta' \in E_\infty^{0,2p^n}$ be an element representing ζ.

Now if $t \geqslant 2p^n + 2$, and if $\mu \in E_t^{r,s}$ with $s = a(2p^n) + b$, then $\mu := (\zeta')^a \mu'$ for some $\mu' \in E_t^{r,b}$. It follows that $d_t(\mu) = (\zeta')^a d_t(\mu') = 0$. That is, ζ' is regular on $E_t^{*,*}$ and ζ is regular on $\mathrm{H}^*(G, k)$. Therefore, $d_t = 0$ for $t \geqslant 2p^n + 2$ as asserted. $\square$

Theorem 3.3. *Let f and n be positive integers. Suppose that*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is an extension of finite p-groups such that

(1) *$|H| \leqslant n$, and*
(2) *G has an abelian subgroup A such that $H \subseteq A$ and $|G : A| \leqslant f$.*

Then the ring $\mathrm{H}^*(G, k)$ is determined up to a finite number of possibilities by the ring $\mathrm{H}^*(Q, k)$.

Proof. Consider first the case that $H \cong C_p$. In that case, the conditions of Proposition 3.1 are satisfied and the cohomology ring $\mathrm{H}^*(G, k)$ is determined by that of $\mathrm{H}^*(Q, k)$ and $\mathrm{H}^*(C_p, k)$ up to a finite number of possibilities. As the cohomology ring $\mathrm{H}^*(C_p, k)$ is known, we say that $\mathrm{H}^*(G, k)$ is determined by $\mathrm{H}^*(Q, k)$ up to a finite number of possibilities. Now the proof of the theorem follows from the fact that we can form G from Q by taking a sequence of at most n extensions by C_p. For each extension there are only finitely many possibilities for the cohomology ring $\mathrm{H}^*(G, k)$, given the cohomology ring of Q. $\square$

The next theorem approaches the extension from another direction, with the cyclic factor on top.

Theorem 3.4. *Suppose that G is an extension $1 \longrightarrow H \longrightarrow G \longrightarrow C_p \longrightarrow 1$ where H is a finite p-group. Suppose we know that the spectral sequence of the extension stops at page m for some fixed m. Then $\mathrm{H}^*(G, k)$ is determined by $\mathrm{H}^*(H, k)$ up to a finite number of possibilities.*

Proof. First note that there is only a finite number of possible actions of C_p on $\mathrm{H}^*(H,k)$ and so it does not matter if we know what that action is. Suppose that we choose a homogeneous set of parameters $\gamma_1, \ldots, \gamma_t$ for the ring of invariants $\mathrm{H}^*(H,k)^{C_p}$ of $\mathrm{H}^*(H,k)$ under the action of C_p. That is, $\gamma_1, \ldots, \gamma_t$ are elements such that the subring that they generate is a polynomial ring and the ring of invariants is finitely generated as a module over the subring. Let $\zeta_i = \gamma_i^p$. Then ζ_i is in the image of the restriction of $\mathrm{H}^*(G,k)$ to $\mathrm{H}^*(H,k)$, by a norm argument (see Theorem 6.3.5 of [3]). Also, $\zeta_1, \ldots, \zeta_t$ is a homogeneous set of parameters for $\mathrm{H}^*(H,k)$. Let S be the subring of $E_2^{*,*}$ generated by the classes of $\zeta_1, \ldots, \zeta_t$ in $E_2^{0,*}$ and by μ where μ is the class of the inflation of the generator in $\mathrm{H}^2(C_p,k)$ to $E_2^{2,0}$. Note that each of the differentials d_n is zero on the classes of $\zeta_1, \ldots, \zeta_t$ and μ on the nth page of the spectral sequence.

The ring $E_2^{*,*}$ is finitely generated as a module over S. Moreover, since $\mathrm{H}^s(H,k)$ is periodic of period at most 2 as a module over $\mathrm{H}^*(C_p,k)$, the generators of $E_2^{*,*}$ can be assumed to lie in $E_2^{0,*} \cup E_2^{1,*}$. The differentials in the spectral sequence are all S-module homomorphisms. It should be clear that the module $E_2^{j,*} = \mathrm{H}^j(C_p, \mathrm{H}^*(H,k))$, as a module over the ring generated by $\zeta_1, \ldots, \zeta_t$, is determined by the action of C_p on $\mathrm{H}^*(H,k)$ and is independent of the actual group H.

Now suppose that $\alpha_1, \ldots, \alpha_w$ is a set of homogeneous generators for $E_2^{*,*}$ as an S-module. For each j, there is only a finite number of choices for $d_2(\alpha_i)$. Hence $E_3^{*,*}$, which is the homology of $E_2^{*,*}$, is determined up to a finite number of choices by $E_2^{*,*}$. Likewise, $E_4^{*,*}$ is determined up to a finite number of choices by $E_3^{*,*}$. Continuing in this way we get that $E_n^{*,*}$ is determined up to a finite number of choices by $\mathrm{H}^*(H,k)$. However, $E_n^{*,*} = E_\infty^{*,*}$. The proof of the theorem is completed by an application of Theorem 2.1. $\square$

Finally, we are interested in extensions from another angle, where we know the cohomology of the extension and we want to construct the cohomology of the subgroup. Once again we are not assuming that we know anything about the group G beyond its cohomology ring.

Theorem 3.5. *Let n be a positive integer. Suppose that S is a finitely generated k-algebra. Then there are only finitely many k algebras R with the property that $R \cong \mathrm{H}^*(H,k)$ for H a subgroup of a p-group G with $\mathrm{H}^*(G,k) \cong S$ and $|G:H| \leqslant p^n$.*

Proof. By an induction we can assume that $n=1$ and that $G/H \cong C_p$ is a cyclic group of order p. In any case, $\mathrm{H}^*(H,k)$ is finitely generated as a module over $\mathrm{H}^*(G,k)$, and in fact it is isomorphic to the module $\mathrm{H}^*(H,k) \cong \mathrm{H}^*(G, k_H^{\uparrow G})$ by the Eckmann-Shapiro Lemma. The induced module $k_H^{\uparrow G}$ has a unique sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_p = k_H^{\uparrow G}$$

such that for each i, M_i has dimension i. Hence we have a collection of exact sequences $0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow k \longrightarrow 0$, and long exact sequences

$$\ldots \longrightarrow \mathrm{H}^r(G, M_{i-1}) \longrightarrow \mathrm{H}^r(G, M_i) \longrightarrow \mathrm{H}^r(G,k) \xrightarrow{\delta} \mathrm{H}^{r+1}(G, M_{i-1}) \longrightarrow \ldots,$$

where the maps commute with multiplication by elements of $\mathrm{H}^*(G, k)$. Now assume by induction that there is only a finite number of possibilities for $\mathrm{H}^*(G, M_{i-1})$ as an $\mathrm{H}^*(G, k)$-module. This is certainly the case for $i = 2$. There are only finitely many possibilities for the map δ. We have an exact sequence of $\mathrm{H}^*(G, k)$-modules

$$0 \longrightarrow \mathrm{Coker}(\delta) \longrightarrow \mathrm{H}^*(G, M_i) \longrightarrow \mathrm{Ker}(\delta) \longrightarrow 0.$$

It remains only to prove that there is only a finite number of possibilities for $\mathrm{H}^*(G, M_i)$. For this we need only show that $\mathrm{Ext}_{\mathrm{H}^*(G,k)}(\mathrm{Ker}(\delta), \mathrm{Coker}(\delta))$ is finite. However, if

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathrm{Ker}(\delta) \longrightarrow 0$$

is a projective resolution of $\mathrm{Ker}(\delta)$, then because $\mathrm{H}^*(G, k)$ is noetherian, we can assume that P_1 is finitely generated. Moreover, any cocycle $P_1 \longrightarrow \mathrm{Coker}(\delta)$ must be a graded map. Hence there are only finitely many choices for the image of each of the generators of P_1. Hence there are only finitely many possibilities for the structure of $\mathrm{H}^*(G, M_i)$ as an $\mathrm{H}^*(G, k)$-module.

It follows from Theorem 2.3 that there are only finitely many possibilities for the ring structure on $\mathrm{H}^*(G, M_p) \cong \mathrm{H}^*(H, k)$. □

4. Root groups

In this section we want to investigate the cohomology rings of some of the groups at the stems of the trees associated to the coclass classification of p-groups. For $p = 2$ these groups are all wreath products of dihedral groups or of extensions of abelian groups by quaternion groups. Several results are already known. We begin by reminding the reader of an old calculation [4].

Lemma 4.1. *If G is a dihedral group of order 2^n for some $n \geqslant 2$, then*

$$\mathrm{H}^*(G, k) \cong k[z, y, x]/(zy),$$

where z and y have degree 1 and x has degree 2.

Let $\mathscr{D}_1$ be the split extension

$$0 \longrightarrow T_1 \longrightarrow \mathscr{D}_1 \longrightarrow C_2 \longrightarrow 1,$$

where $T_1 \cong \mathbb{Z}_2$ is a copy of the 2-adic integers, and C_2 acts on T_1 by inverting elements (multiplication by -1). Let $\mathscr{D}_{1,j} = \mathscr{D}_1/2^j T_1$. Then $\mathscr{D}_{1,j}$ is a dihedral group of order 2^{j+1}.

Let $\mathscr{D}_2 = \mathscr{D}_1 \wr C_2$ be the wreath product of a $\mathscr{D}_1$ with a cyclic group of order 2. More generally, let $\mathscr{D}_n = \mathscr{D}_{n-1} \wr C_2$. Then $\mathscr{D}_n$ is a uniserial 2-adic space group with translation group $T_n \cong T_1^{2^{n-1}}$ and point group $P_n \cong C_2 \wr C_2 \wr \cdots \wr C_2$. Let $\mathscr{D}_{n,j} = \mathscr{D}_n/2^j T_n$. The following should be obvious.

Lemma 4.2. *For $n \geqslant 2$ and $j \geqslant 1$, we have that $\mathscr{D}_{n,j} \cong \mathscr{D}_{n-1,j} \wr C_2$.*

Proposition 4.3. *For all* $n \geqslant 1$ *and* $i, j \geqslant 2$, *we have that* $\mathrm{H}^*(\mathscr{D}_{n,i}, k) \cong \mathrm{H}^*(\mathscr{D}_{n,j}, k)$ *where the isomorphism is an isomorphism of rings.*

Proof. The proposition is a straightforward application of Nakaoka's Theorem [3, Theorem 6.2.4] which says that the cohomology ring of a wreath product such as $\mathscr{D}_{n,j} \cong \mathscr{D}_{n-1,j} \wr C_2$ is isomorphic as a ring to the the E_2 page of the spectral sequence of the group extension

$$1 \longrightarrow \mathscr{D}_{n-1,j} \times \mathscr{D}_{n-1,j} \longrightarrow \mathscr{D}_{n,j} \longrightarrow C_2 \longrightarrow 1.$$

Hence the result is a consequence of Lemma 4.1 and repeated application of Lemma 4.2 and Nakaoka's Theorem. □

There is one other set of space groups that we must consider. These are slightly more complicated. Let $\mathscr{Q}_1$ be the split extension

$$1 \longrightarrow T_1 \longrightarrow \mathscr{Q}_1 \longrightarrow Q_{16} \longrightarrow 1,$$

where Q_{16} is a quaternion group of order 16 and $T_1 \cong \mathbb{Z}_2^4$ is a sum of four copies of the 2-adic integers. The action of Q_{16} on T_1 is given in the paper [9]. The details of that action are not really important to us here. For $j \geqslant 1$, let $\mathscr{Q}_{1,j} = \mathscr{Q}_1/2^j T_1$. As before, for $n > 1$ we let $\mathscr{Q}_n = \mathscr{Q}_{n-1} \wr C_2$ and $\mathscr{Q}_{n,j} = \mathscr{Q}_n/2^n T_n$ where $T_n \cong \mathbb{Z}_2^{2^{n+1}}$ is the translation group of $\mathscr{Q}_n$. It is straightforward to prove the following.

Lemma 4.4. *For* $n \geqslant 2$ *and* $j \geqslant 1$, *we have that* $\mathscr{Q}_{n,j} \cong \mathscr{Q}_{n-1,j} \wr C_2$.

The first result we want to prove is the following. Among other things, the proposition implies that there is an infinite collection of the groups $\mathscr{Q}_{1,j}$ with isomorphic mod-2 cohomology rings.

Proposition 4.5. *There are only finitely many possible cohomology rings of the form* $\mathrm{H}^*(\mathscr{Q}_{1,j}, k)$ for all $j \geqslant 2$.

The proof requires two steps. The first we state as a lemma.

Lemma 4.6. *Let S be the split extension*

$$0 \longrightarrow T_1 \longrightarrow S \longrightarrow C_8 \longrightarrow 0,$$

where C_8 *is the maximal subgroup of order* 8 *in* Q_{16}. *Thus, S is a subgroup of index* 2 *in* $\mathscr{Q}_1$. *Let* $S_j = S/2^j T_1$. *There are only finitely many cohomology rings of the form* $\mathrm{H}^*(S_j, k)$ *for all j.*

Proof. The important thing is that $S \subseteq \mathscr{D}_3$ and moreover, $|\mathscr{D}_3 : S| = 2^4 = 16$. The point is that the element of order 8 that generates the cyclic group at the point group of S is conjugate to an element of order 8 in the point group of $\mathscr{D}_3$, whose translation group is also $\mathbb{Z}_2^4$. Now by Theorem 3.5 on subgroups, $\mathrm{H}^*(S_j, k)$ is determined by $\mathrm{H}^*(\mathscr{D}_{3,j}, k)$ up to a finite number of possibilities. But now $\mathrm{H}^*(\mathscr{D}_{3,j}, k) \cong \mathrm{H}^*(\mathscr{D}_{3,i}, k)$ for all i and j. Hence $\mathrm{H}^*(S_j, k)$ is determined by $\mathrm{H}^*(\mathscr{D}_{3,1}, k)$ up to a finite number or possibilities. □

Proof of Proposition 4.5. We want to consider the spectral sequence of the group extension

$$1 \longrightarrow S_j \longrightarrow \mathcal{Q}_{1,j} \longrightarrow C_2 \longrightarrow 1$$

which has the form $E_2^{r,s} = \mathrm{H}^r(C_2, \mathrm{H}^s(S_j, k)) \Rightarrow \mathrm{H}^{r+s}(\mathcal{Q}_{1,j}, k)$. By Theorem 3.4, the cohomology of $\mathcal{Q}_{1,j}$ is determined up to a finite number of possibilities by the cohomology ring of S_j provided we can show that the spectral sequence stops after some fixed number of pages, where that number does not depend on j. Consequently, we are done once we show that the spectral sequence stops.

For the spectral sequences, note that $E_2^{1,0} = E_\infty^{1,0}$ is generated by an element η that is inflated from a generator of $\mathrm{H}^1(C_1, k)$. Indeed, we know that for any $r > 0$, $E_2^{r,s} = \eta E_2^{r-1,s}$. The inflation from C_2 to $\mathcal{Q}_{1,j}$ factors through the inflation $\mathrm{H}^*(C_2, k) \longrightarrow \mathrm{H}^*(Q_{16}, k)$ where $Q_{16} \cong \mathcal{Q}_1/T_1$. It follows that $\eta^3 = 0$ in $E_4^{*,*}$ because this is what happens in the cohomology of Q_{16}. The implication is that $E_4^{r,s} = 0$ for $r > 3$ and hence that $E_4^{r,s} = E_\infty^{r,s}$ for all r and s. This completes the proof of the proposition. □

Proposition 4.7. *For any n, there is only a finite number of possible cohomology rings of the form* $\mathrm{H}^*(\mathcal{Q}_{n,j}, k)$ *for all* $j \geqslant 2$.

Proof. This follows the same ideas as the proof of Proposition 4.3. That is, for each n the cohomology of $\mathcal{Q}_{n,j} \cong \mathcal{Q}_{n-1,j} \wr C_2$ is determined entirely from the cohomology ring $\mathrm{H}^*(\mathcal{Q}_{n-1,j}, k)$, using the spectral sequence. Inductively, we see that there is only a finite number of possibilities for $\mathrm{H}^*(\mathcal{Q}_{n,j}, k)$ by Proposition 4.5. □

5. The main theorem

Suppose that G is a p-group. The purpose of this section is to show that the cohomology ring of G is determined up to a finite number of choices by the coclass of G. The main theorem is the following.

Theorem 5.1. *Let k be a field of characteristic* 2. *For any natural number r there are only finitely many graded commutative k-algebras R with the property that* $R \cong \mathrm{H}^*(G, k)$, *where G is a* 2*-group of coclass r*.

Hence the theorem says that among all of the groups G of coclass r there is only a finite number of cohomology rings $\mathrm{H}^*(G, k)$ up to isomorphism. The proof of the theorem relies very heavily on the classification of p-groups by coclass [7]. The classification can be expressed as follows. Note that we are identifying a group with its isomorphism class.

Theorem 5.2. *Let r be a positive integer and let* $\mathcal{G}_r$ *be the collection of all* 2*-groups of coclass r. Then* $\mathcal{G}_r = \mathcal{G}'_r \cup \mathcal{G}''_r$ *where* $\mathcal{G}'_r$ *and* $\mathcal{G}''_r$ *satisfy the following properties*:

(1) $\mathcal{G}''_r$ *has only a finite number of elements*.

(2) *If G is in* $\mathscr{G}'_r$, *then there exist a normal subgroup* $N \subseteq G$, *a uniserial* 2*-adic space group S of coclass r and a normal subgroup A of S such that*

(a) *A is contained in the translation subgroup of S*,
(b) $|N| \leqslant 2^{2^{r-1}(2^{r-1}+r+3)}$, *and*
(c) $G/N \cong S/A$.

Proof. The theorem is a rewrite of Theorem 7.6 of [7], except that we have incorporated a definition of "constructible" (see the paragraph preceding Lemma 5.1 of [7]). □

Recall that a 2-adic space group is a group S that has a normal subgroup $T \cong \mathbb{Z}_2^t$ for some t, such that $B = S/T$ is a finite 2-group. The action of B on T should be irreducible. In particular, for each n there is one and only one P-invariant subgroup of T of index 2^n in T where P is the point group of S. That subgroup is an element of the lower central series for S. So we have an exact sequence

$$0 \longrightarrow T \longrightarrow S \longrightarrow B \longrightarrow 1.$$

The subgroup T is called the translation subgroup and B is called the point group. In the examples of the last section, for $\mathscr{D}_n$, the translation subgroup is $\mathbb{Z}_2^{2^{n-1}}$ while the point group is $C_2 \wr \cdots \wr C_2$ (n copies), and for $\mathscr{Q}_n$ the translation group is $\mathbb{Z}_2^{2^{n+1}}$ and the point group is $Q_{16} \wr C_2 \wr \cdots \wr C_2$. The result we want is the following.

Theorem 5.3. *Suppose that S is a* 2*-adic uniserial space group with translation subgroup* $T \cong \mathbb{Z}_2^{2^{n-1}}$ *and point group P. Then either* $S \subseteq \mathscr{D}_n$ *or* $S \subseteq \mathscr{Q}_{n-2}$. *That is, S is isomorphic to a subgroup of either* $\mathscr{D}_n$ *or* $\mathscr{Q}_{n-2}$. *Moreover, the embedding is such that T is a normal subgroup of the translation group of* $\mathscr{D}_n$ *or* $\mathscr{Q}_{n-2}$ *as appropriate.*

Proof. The proof can be found in the literature, but does not appear to be succinctly written in any one location. For that reason, we present a brief sketch. As in [8, p. 76] we can blow up T to $V \cong T \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$, yielding a split extension X of V by the point group P. Now P is conjugate to a subgroup Q of the point group of either D_n or Q_{n-2} (see [10] or [9]). So assume that $P \subseteq Q$ and let $\tilde{T}$ be a Q-invariant superlattice

$$T \subseteq \tilde{T} \subseteq V \cong \mathbb{Q}_2^{2^{n-1}}.$$

We can assume that $\tilde{T}$ is chosen so that the extension R, given as

$$0 \longrightarrow \tilde{T} \longrightarrow R \longrightarrow Q \longrightarrow 1$$

is isomorphic to either $\mathscr{D}_n$ or $\mathscr{Q}_{n-2}$. Thus, $S \subseteq R$, where both are regarded as subgroups of X.

As to the invariance of T, we note that by the uniseriality assumption, T is the only P-invariant subgroup of $\tilde{T}$ having index $|\tilde{T} : T|$. Likewise, there is one and only one Q-invariant sublattice of $\tilde{T}$ having that index, and the sublattice must be T. □

In considering coclass we need the following. It is *Conjecture E* of [7] and was first proved in [9], though it also follows from the alternate approach to the subject presented in [12].

Theorem 5.4. *For any $r > 0$, there are only finitely many isomorphism classes of 2-adic uniserial space groups of coclass r.*

Lemma 5.5. *Let r be a positive integer. There exist integers $\gamma(r)$ and $\delta(r)$ with the following properties. Suppose that G is a finite 2-group in $\mathscr{G}'_r$ and that the translation subgroup of the 2-adic uniserial space group associated to G in Theorem 5.2(2) has rank 2^{n-1}. Then G has a normal subgroup N such that*

(1) $|N| \leqslant \gamma(r)$, *and*
(2) *G/N is isomorphic to a subgroup of a group Q where either $Q = \mathscr{D}_{n,j}$ or $Q = \mathscr{Q}_{n-2,j}$ for some j, and where $|Q : G/N| \leqslant \delta(r)$.*

Proof. By Theorem 5.2, there exists a 2-adic uniserial space group S of coclass r, a subgroup A of the translation subgroup of S and a normal subgroup N' of G such that $G/N' \cong S/A$. In turn, S is a subgroup of a space group R such that $R \cong \mathscr{D}_n$ or $R \cong \mathscr{Q}_{n-2}$ and A is a normal subgroup of the translation subgroup T of R. Therefore, G/N' is isomorphic to a subgroup of R/A and its index in R/A is the same as the index of S in R. Note that because there are only finitely many 2-adic uniserial space groups S of coclass r, we may assume that the index $|R/A : G/N'|$ is bounded by a number B that depends only on the coclass r, and not on the particulars of G or N'.

Suppose that T is the translation subgroup of R. Then for some j,

$$2^{j+1}T \subseteq A \subseteq 2^jT \quad \text{and} \quad |2^jT : A| < 2^n.$$

Let N be the kernel of the composition $G \longrightarrow R/A \longrightarrow R/(2^jT)$. Note here that $R/(2^jT)$ is isomorphic to either $\mathscr{D}_{n,j}$ or to $\mathscr{Q}_{n-2,j}$. Moreover, the order of N/N' is at most 2^n while the index $|R/(2^jT) : G/N|$ is at most B. Finally by Theorem 5.2, the order N' is bounded by $2^{2^{r-1}(2^{r-1}+r+3)}$. So the theorem is proved by letting $\delta(r)$ be B and letting γr be $2^u 2^{2^{r-1}(2^{r-1}+r+3)}$ where 2^u is the largest rank of the translation subgroup of any uniserial 2-adic space group of coclass r. $\square$

With this result in mind, we can prove the main theorem.

Proof of Theorem 5.1. Because the collection $\mathscr{G}''_r$ is finite, it is only necessary to show that there are only finitely many rings of the form $\mathrm{H}^*(G,k)$ for G a 2-group in $\mathscr{G}'_r$. For $G \in \mathscr{G}'_r$, there is a normal subgroup $N \subseteq G$ such that G/N is isomorphic to a subgroup of either $\mathscr{D}_{n,j}$ or $\mathscr{Q}_{n-2,j}$ for some n and some j. Note that n is the rank of the translation group of some uniserial 2-adic space group of coclass r. Hence n is bounded. Therefore by Propositions 4.3 and 4.7 there are only finitely many possible cohomology rings for $\mathscr{D}_{n,j}$ or $\mathscr{Q}_{n-2,j}$. Then by Theorem 3.5, there are only finitely many possibilities for the cohomology ring $\mathrm{H}^*(G/N,k)$. Finally, by Proposition 3.3 and the bound on the order of N in Lemma 5.5, we are finished. $\square$

6. Problems and remarks

It seems difficult to believe that the Main Theorem 5.1 is not also true in the case of p-groups for p an odd prime. That is, it would seem that for any prime p and integer r, there should be only a finite number of cohomology ring $\mathrm{H}^*(G,k)$ (up to isomorphism) for all p-groups G of coclass r and any field k of characteristic p. Unfortunately, the proof given in the last section does not work for odd p. There are two reasons.

The first difficulty is that for odd primes p, the classification is slightly different. In the case that p is odd we must allow for the possibility that G/N is only isomorphic to a twisted version of S/A in part (2) of Theorem 5.2. That is, suppose T is the translation subgroup of S. Let B be a subgroup of T that contains A. It is possible to form nonabelian central extensions

$$0 \longrightarrow B/A \longrightarrow H \longrightarrow T/B \longrightarrow 1,$$

where the cocycle defining the extension is only altered from that defining T/A by addition of nontrivial commutators. Then, in the classification of p-groups for odd p, the collection $\mathscr{G}'_r$ include groups G with $G/N \cong W$ where W is an extension of the form

$$0 \longrightarrow B/A \longrightarrow W \longrightarrow S/B \longrightarrow 1$$

that is determined by cocycles as above. See Section 5 of [7] for more explicit details. The point is that proof of Theorem 5.1 for p-groups with p odd would have to accommodate this difference in the classification. However, this part can be easily handled by methods that are similar to those in the proof of Theorem 3.5. The other difficulties may be more fundamental.

The other problem with proving an analog of Theorem 5.1 in odd characteristics is that there is no analog to Propositions 4.3 and 4.7. What is true for odd p is that every uniserial p-adic space group of coclass r is a subgroup of finite index in one of the group R_n which can be described as follows (see [8]). Let T_1 be the free $\mathbb{Z}_p$-lattice of rank $p-1$ on which C_p acts by the companion matrix to the polynomial $x^{p-1}+\cdots+x+1$. As a $\mathbb{Z}_pC_p$-module it can be regarded as the kernel of the augmentation map $\mathbb{Z}_pC_p \longrightarrow \mathbb{Z}_p$. Let R_1 be the split extension

$$0 \longrightarrow T_1 \longrightarrow R_1 \longrightarrow C_p \longrightarrow 1$$

with the given action of C_p. Then inductively, let $R_{n+1} = R_n \wr C_p$. One approach to a proof of Theorem 5.1 in odd characteristics would be to verify an analog of Proposition 4.5 for the groups $R_{1,j} = R_1/(p_j T_1)$. So we are left with the following question.

Question 6.1. Let p and a field k of characteristic p be given. Is it possible that all of the cohomology rings $\mathrm{H}^*(R_{1,j},k)$ are isomorphic?

There is a tiny amount of experimental evidence to support a conjecture in this direction. Informally, David Green, at the authors suggestion, computed the cohomology rings for $p=3$ and $j=1,2$, using his own implementations of cohomology programs [6], and the rings appear to be isomorphic.

Because every p-group of coclass r must have a metabelian subgroup of bounded index, another approach might be to attempt to answer the following question.

Question 6.2. Suppose that we have an extension of the form

$$1 \longrightarrow H \longrightarrow G \longrightarrow C_p \longrightarrow 1,$$

where H is a p-group. Is the cohomology ring of G (over a field of k characteristic p) determined up to a finite number of possibilities by the cohomology ring of H?

It turns out that this is the same question as asking if the spectral sequence of the extension must stop after a certain number of steps, where that number depends only on p and the cohomology ring $\mathrm{H}^*(H, k)$.

As noted in the introduction, there is some experimental evidence to suggest that the cohomology rings of successive quotients of the 2-adic uniserial space groups have isomorphic cohomology rings more often than can be justified using the methods of the proof of Theorem 5.1. The 2-adic uniserial space groups of coclass 1, 2 and 3 were enumerated by Newman and O'Brien in [11]. O'Brien has also provided the author with the data base suitable for input into MAGMA for the 2-adic uniserial space groups of coclass 1, 2 and 3. The quotients $R/\gamma_i(R)$, where R is a 2-adic unserial space group and $\gamma_i(R)$ is the ith term in its lower central series, can be obtained using the ANU p-quotient algorithm.

The cohomology rings $\mathrm{H}^*(G, k)$ for $G = G_{R,i} \cong R/\gamma_i(R)$ and $k = \mathbb{F}_p$ were computed for several of the families of 2-groups and for some values of i using the authors cohomology ring programs. In a surprisingly large number of cases, it happens that $\mathrm{H}^*(G_{R,i}, k)$ did not depend on the value of i as long as i was sufficiently large. In particular, to have any meaning, the order of $G_{R,i}$ should be at least 2^{q+c} where 2^q is the order of the point group of R and c is the rank of the translation subgroup of R. The computer was unable to handle the calculations of the cohomology rings for groups of order larger than 256, and could compute the cohomology of the larger groups only if the degrees of the cohomology generators were not too large. Hence the body of evidence is not huge. But here is a sampling of what we can say.

For the family number 23 in the scheme of [11] the groups of order 32, 64, 128, and 256 all have isomorphic mod-2 cohomology rings. In this case, the translation subgroup T has rank 2 and R/T has order 8. Because the rank is 2, it might be expected that the two quotients of orders 64 and 256 would have isomorphic cohomology rings, and likewise for the quotients of order 32 and 128. However, the cohomology rings are all isomorphic to each other. In this case, the cohomology rings have the following form. Let $P = k[z, y, x, w, v, u, t, s]$, where the variables are in degrees 1, 1, 2, 2, 2, 3, 3, 4. Then $\mathrm{H}^*(G, k) \cong P/\mathscr{I}$ where $\mathscr{I}$ is the ideal generated by the elements

$$zy,\ y^2,\ z^3 + yx,\ zx + yx,\ yw,\ z^2v + zt,$$

$$x^2,\ yu,\ yt,\ z^2u + z^2t + xt,\ xu,$$

$$z^2w^2 + zvt + t^2,\ xw^2 + zvu + zvt + ut + t^2,$$

$$zwu + zvu + zvt + z^2s + u^2 + ut + t^2.$$

We should note that the computer did not obtain exactly the same relations for cohomology rings in the different cases. Instead, the isomorphisms were established by making changes of variables.

For the family number 21, it also appears that the cohomology rings for the groups of order 32, 64, 128, and 256 are isomorphic. In this case, the rings have 11 generators and 34 relations. The relations were sufficiently complicated that it would be very tedious to verify an actual isomorphism. However, the rings are very similar. In this case $\mathrm{Rank}(T) = 2$ and $|R/T| = 8$. A similar thing happened for the groups of order 64. 128 and 256 for family number 24 ($\mathrm{Rank}(T) = 2$ and $|R/T| = 16$). In other cases it was not difficult to be more definite. The groups of order 32, 64, 128 and 256 had isomorphic cohomology rings for the families with numbers 22 ($\mathrm{Rank}(T)=2$ and $|R/T|=8$), 25 ($\mathrm{Rank}(T)=1$ and $|R/T|=8$), 27 ($\mathrm{Rank}(T)=1$ and $|R/T|=8$), 47 ($\mathrm{Rank}(T)=1$ and $|R/T|=8$), and 48 ($\mathrm{Rank}(T)=1$ and $|R/T|=8$). For family number 63 ($\mathrm{Rank}(T) = 2$ and $|R/T| = 8$) the groups of order 64 and 128 have isomorphic cohomology rings.

In all of the cases above we looked only at the quotients $G = R/\gamma_i(R)$ for R a 2-adic uniserial space group. There is also the question of what happens to isomorphic cohomology rings under extension that have the same extension class in $\mathrm{H}^2(G, k)$. Do we obtain isomorphic cohomology rings?

We end with a remark about the Steenrod algebra. The cohomology ring $\mathrm{H}^*(G, k)$ is an unstable module over the Steenrod algebra and we can ask the question of whether there are only finitely many algebras over the Steenrod algebra that can be the cohomology rings of 2-groups of fixed coclass r. The answer is yes and the proof follows directly from the Main Theorem 5.1. That is, the Cartan formula for the action of the Steenrod algebra implies that the action is determined entirely by the action on the generators of $\mathrm{H}^*(G, k)$. Since the Steenrod algebra is defined over the prime field $\mathbb{F}_2$, there are only finitely many choices that can be made for the image of any of the Steenrod operators on a generator for the cohomology ring. Of course, the Steenrod algebra is infinitely generated, but the operators in high enough degree must annihilate all of the generators for the cohomology ring. Hence there is only a finite number of choices to be made.

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